Solutions

EXAM III
MATH 142 -CALCULUS II

There are six questions each worth 10 points.
Carefully read the instruction at the top of each page.
This is a closed book exam - you are not permitted to use a notecard or any notes, besides the trigonometric identity sheet. If you would like extra scratch paper raise your hand. Calculators and other electronic aides are not permitted.

Good luck!

Question 1

Find a Taylor (or Maclaurin) expansion of $\frac{-2}{4+x}$. You may center the expansion at any point x_0 however I recommend using a geometric series. Find the Radius of Convergence and the Interval of Convergence. Determine if the endpoints of the interval converge, converge absolutely or diverge.

$$\frac{-2}{4+x} = \frac{-2}{4(1+\frac{x}{4})} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n.$$

This geometric series converges when $\left|-\frac{x}{4}\right|<1$ or $|x|<4$.

Thus, the radius of convergence is $R=4$ and the interval of convergence is $I=(-4,4)$.

At $x=-4$, $-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{4}{4}\right)^n = -\frac{1}{2}\sum_{n=0}^{\infty} 1$ diverges.

At $x=4$, $-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-4}{4}\right)^n = -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n$ diverges.

Question 2

Find the Taylor expansion of $\cos(2x)$ centered at $x_0 = \pi$. Find the Radius of Convergence and the Interval of Convergence.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos(2x)$	1
1	$-2\sin(2x)$	0
2	$-4\cos(2x)$	-4
3	$8\sin(2x)$	0
4	$16\cos(2x)$	16
$\vdots$	$\vdots$	$\vdots$
$n=2k$	$(-1)^k 2^{2k}\cos(2x)$	$(-1)^k 2^{2k}$
$n=2k+1$	$(-1)^{k+1} 2^{2k+1}\sin(2x)$	0

Thus

$$\cos(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k}}{(2k)!}(x-\pi)^{2k}.$$

Using the ~~[illegible]~~ Ratio test,

$$\lim_{k\to\infty}\left|\frac{\frac{2^{2k+2}}{(2k+2)!}(x-\pi)^{2k+2}}{\frac{2^{2k}}{(2k)!}(x-\pi)^{2k}}\right| = \lim_{k\to\infty}\left|\frac{2^{2k+2}(x-\pi)^{2k+2}}{2^{2k}(x-\pi)^{2k}}\cdot\frac{(2k)!}{(2k+2)!}\right| = \lim_{k\to\infty}\left|\frac{4(x-\pi)^2}{(2k+2)(2k+1)}\right|$$

$$= 0.$$

0 is always less than 1, so the Radius of Convergence, $R=\infty$, and the Interval of Convergence $I=(-\infty,\infty)$.

Question 3

Find the length of the curve parametrized by

$$x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}.$$

(**Hint:** You may need to use trig substitution to evaluate this integral!)

$$\text{Length} = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

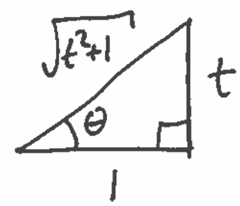
$$= \int_0^{\sqrt{3}} \sqrt{(3t^2)^2 + (3t)^2}\, dt = \int_0^{\sqrt{3}} \sqrt{9t^2(t^2+1)}\, dt$$

$$= \int_0^{\sqrt{3}} 3|t|\sqrt{t^2+1}\, dt.$$ Using trig sub, let $t = \tan\theta$

$dt = \sec^2\theta\, d\theta$

$(0 \le \theta \le \frac{\pi}{2})$

$$= 3\int_{\square}^{\square} \tan\theta \sqrt{\tan^2\theta + 1} \cdot \sec^2\theta\, d\theta$$



$$= 3\int_{\square}^{\square} \tan\theta \cdot \sec^3\theta\, d\theta$$

Let $u = \sec\theta$

$du = \sec\theta \tan\theta\, d\theta$

$$= 3\int_{\square}^{\square} u^2\, du = u^3\Big|_{\square}^{\square} = \sec\theta\Big|_{\square}^{\square}$$

$$= \frac{\sqrt{t^2+1}}{1}\Bigg|_0^{\sqrt{3}} = (2-1) = 1.$$

Question 4

Find the degree 4 Taylor polynomial of the function $e^x \sin x$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

Thus

$$e^x \sin x = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \cdots\right)$$

$$= \left(x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \cdots\right) - \left(\frac{x^3}{6} + \frac{x^4}{6} + \cdots\right)$$

$$= x + x^2 + \frac{x^3}{3} + \cdots.$$

So

the 4th degree Taylor polynomial $P_4(x) = x + x^2 + \frac{x^3}{3}$.

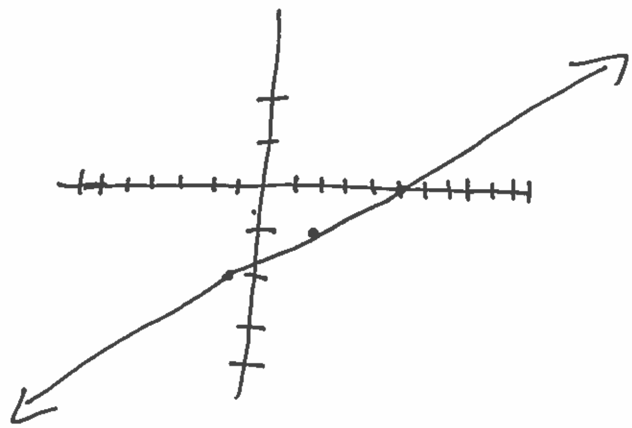
Question 5

Describe the parametrized curve of $x = 3t + 2$, $y = t - 1$ $(-\infty < t < \infty)$ by plotting the curve or by converting to rectangular coordinates. Find dy/dx.

Plotting

t	(x,y)
-3	(-7,-4)
-2	(-4,-3)
-1	(-1,-2)
0	(2,-1)
1	(5,0)
2	(8,1)
3	(11,2)

This is linear; i.e. a line.



or by converting to rectangular coordinates

$y = t - 1 \Rightarrow t = y + 1$. So $x = 3(y+1) + 2 = 3y + 5$.

Again, this is clearly a line.

Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{3} \quad \text{for all } t \in \mathbb{R}.$$

Question 6

Estimate the definite integral $\int_0^1 \cos x^2 dx$ with an error of less than 0.001 using a Taylor Series. You may do as you like (as long as you justify your answer) however I recommend the following steps.

a. Write the Maclaurin Series for $\cos x$ and replace the x with x^2.

b. Integrate the above series to get a series for $\int \cos x^2 dx$.

c. Evaluate the above integral at the limits of integration and determine how many terms are needed to achieve an error of less than 0.001.

(a) $$\cos x^2 = \sum_{k=0}^{\infty} \frac{(-1)^k (x^2)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k}}{(2k)!}$$

(b) $$\int \cos x^2 dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(4k+1)(2k)!} + C$$

$$= C + x - \frac{x^5}{10} + \frac{x^9}{9 \cdot 24} - \frac{x^{13}}{13 \cdot 6!} + \cdots$$

(c) $$\int_0^1 \cos x^2 dx = \sum_{k=0}^{\infty} \frac{(-1)^k x^{4k+1}}{(4k+1)(2k)!} \Bigg|_0^1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)(2k)!}.$$

Using the Alternating Series Error Estimations, we have

$|R_k| \leq |a_{k+1}|$. As $|R_3| \leq \frac{1}{13 \cdot 6!} < 0.001$,

we have that $\int_0^1 \cos x^2 dx \approx 1 - \frac{1}{10} + \frac{1}{9 \cdot 24} = \frac{9^2 \cdot 24 - 10}{10 \cdot 9 \cdot 24}$ with error less than 0.001.